A finite difference approximation of fractional differential equation: errors and oscillations

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Introduction

Fractional differential equations (FDE) have gained much attention in recent decades among researchers in describing natural phenomena found in various physical, engineering and biological applications[2]. They provide more realistic results particularly for stochastic problems. In this work, we consider three different definitions of the fractional derivative with the fractional order α, $0<\alpha<1$. These three definitions are the Riemann-Liouville (RL), Caputo and Grünwald-Letnikov (GL) derivatives [3]. The main objective of this paper is to study how these three definitions perform differently when the low order finite difference approximation is used for the discrete formulation in solving the given FDE for discontinuous equations (Eq.).

Fractional Derivatives

Fractional derivative is defined by the integer order derivative of fractional integral. The followings are three definitions of fractional derivative of order $\alpha, 0<\alpha<1$.

Definition (Riemann-Liouville derivative):

$$_{RL}D^{\alpha}_{a,t}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t(t-\tau)^{n-\alpha-1}f(\tau)d\tau.$$

Definition (Grünwald-Letnikov derivative):

$$_{GL}D^{\alpha}_{a,t}f(t)=\lim_{\substack{h\to0\\Nh=t-a}}h^{-\alpha}\sum_{j=0}^{N}(-1)^j\binom{\alpha}{j}f(t-jh).$$

Definition (Caputo derivative):

$$_{C}D^{\alpha}_{a,t}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau.$$

FD Approximations

We consider finite difference approximations of the initial value α-order FDE for $u(x)$ below

$$\frac{d^{\alpha}u}{dx^{\alpha}}=f(u,x),\quad x=(0,1]. \tag{1}$$

Let v be the numerical approximation of u.

Use $0<\alpha<1$. Suppose that the exact solution $u(x)=x^2$ and the initial condition $u(0)=u_0=0$. The function $f(u,x)$ is

$$f(x)=\frac{2x^{(2-\alpha)}}{\Gamma(3-\alpha)}.$$

Suppose that we divide the interval of $[0,1]$ into N sub-intervals uniformly. Let h be the length of the sub-interval, $h=1/N=\Delta x$.

Finite difference approximation: In each interval each fractional derivative is approximated by the low-order finite difference approximation, i.e. the linear approximation.

Error Analysis

Global errors $e(N)=u(x_N)-v(x_N)$, $x_N=1$:

- Riemann-Liouville derivative - 1st order and α-independent (Left figure of Fig. 1)

$$|e_N|\sim O(h). \tag{2}$$

- Grüwald-Letnikov derivative - 1st order but α-independent (Middle figure of Fig. 1)

$$|e_N|\sim O(h). \tag{3}$$

- Caputo derivative - $2-\alpha$ order (α-dependent) (Right figure of Fig. 1)

$$|e_N|\sim O(h^{2-\alpha}). \tag{4}$$

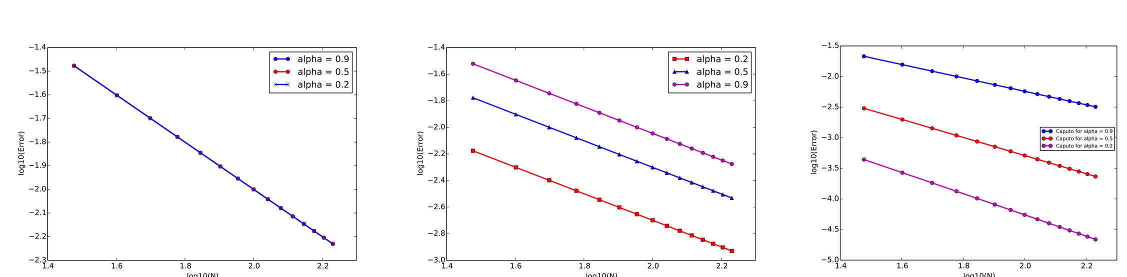

Figure 1: Convergence of errors for $u=x^2$. Left: RL method. Middle: GL method. Right: Caputo method.

- Comparisons of all derivatives (Fig. 2): Among all, the Caputo performs best.

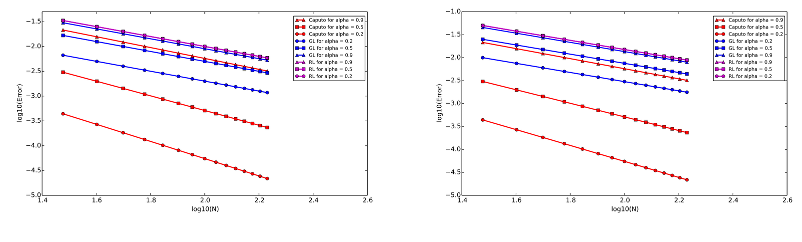

Figure 2: Convergence. Left: $u=x^2$. Right: $u=x^3$.

FDE for Discontinuous Eq.

- Case I: $f(x)$ is discontinuous, e.g. the Heaviside function

$$\frac{d^{\alpha}}{dx^{\alpha}}u=f(x)=H(x-1/2), \tag{5}$$

where $f(x)=0$ for $0<x\le0.5$ and $f(x)=1$ for $x>0.5$.

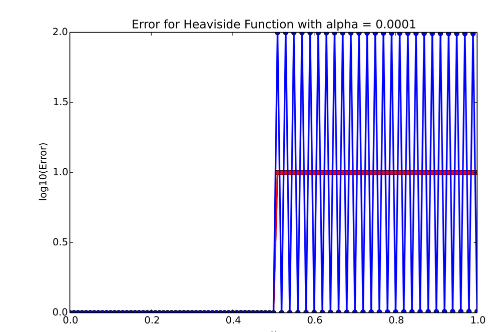

Figure 3: Solutions by the Caputo (black), GL (red) and RL (blue) methods for $\alpha=0.0001$ and $N=100$.

As Figure 3 shows, the solution by the GL and Caputo methods match the exact solution. However, the solution by the RL method is oscillatory as

$$E_j=-E_{j-1}-u'(x_{j-1})\cdot\Delta x.$$

- Case II: $f(x)$ is singular as

$$\frac{d^{\alpha}}{dx^{\alpha}}u=\delta(x-\frac{1}{2})=f(x), \tag{6}$$

where $\delta(x-\frac{1}{2})$ is the Dirac-delta function: $\delta(x-\frac{1}{2})=0$ if $x\neq\frac{1}{2}$ and $\int_{-1}^{1}\delta(x-1/2)dx=1$.

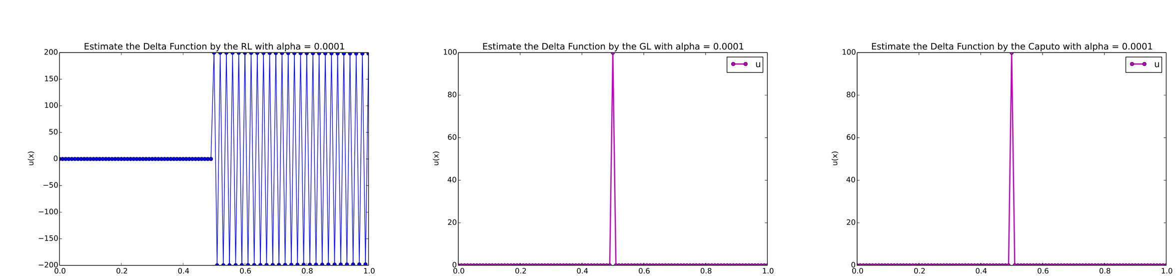

Figure 4: Solutions to Dirac delta functions with $\alpha=0.0001$, $N=100$. Left: RL. Middle: GL. Right: Caputo.

As Figure 4 shows the solution by the GL and Caputo methods converge to the exact solution. However, the solution by the RL method is oscillatory between $\sim\pm2N$.

- Case III: $f(x)$ is smooth as

$$\frac{d^{\alpha}}{dx^{\alpha}}u=f(x)=\sin(2\pi x). \tag{7}$$

As Figure 5 shows the solution by the Caputo converges to the exact solution but the RL method is oscillatory as

$$E_j=-E_{j-1}-2\pi\cdot\Delta x\cdot\sin(2\pi x_{j-1}).$$

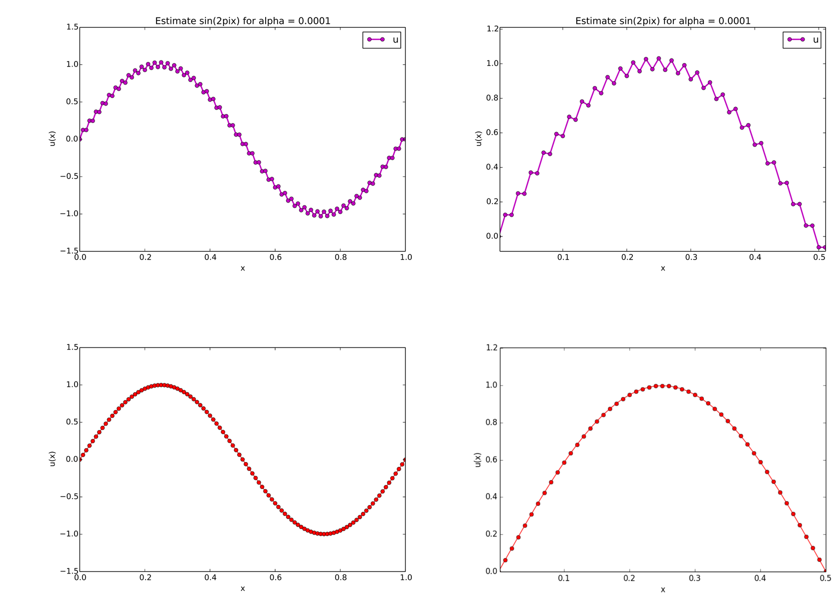

Figure 5: Left: Solutions with $f(x)=\sin(2\pi x)$ with $\alpha=0.001$. Right: Zoomed profile. Top: RL method. Bottom: Caputo method.

Summary and Future Work

- The finite difference approximation with the Caputo derivative yields the best result among others.
- The Riemann-Liouville derivative yields the result that is independent of the fractional order, α.
- The solution with the Riemann-Liouville derivative is oscillatory as $\alpha\to0$ while the solution is stable. For discontinuous problems, the oscillations are significant for the RL method.
- The Caputo and Grünwald-Letnikov derivatives yield non-oscillatory solutions even though $\alpha\to0$.
- In the future work, we will further study the FDE for discontinuous problems with high order finite difference method.

References

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[3] Podlubny I. Fractional differential equations. San Diego: Academic press; 1999.